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AN INVERTABLE 3-DIAGRAM WITH 8 VERTICES

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We present a 3-diagram $\mathscr{D}$ with 8 vertices, which is not isomorphic to a Schlegel-diagram of a 4-polytope, and which is invertable, i.e. it may be realized with any of its 3-cells as the basic facet. A slight modification of $\mathscr{D}$ yields a diagram $\mathscr{D}'$, which cannot be realized by a polyhedral sphere.

0. Introduction

The boundary complex $\mathscr{B}(P)$ of a (convex) d-polytope $P \subset E^d$ may be projected into any of its facets ($(d-1)$-faces). Such a representation of $\mathscr{B}(P)$ in $(d-1)$-space is called a *Schlegel-diagram* of P (see [6]). A Schlegel-diagram is a special case of a more general object, called d-diagram. A finite family $\mathscr{D} = \{D_0\} \cup \mathscr{C}$ of polytopes in E^d shall be called a *d-diagram* provided

(i) $\mathscr{C}$ is a geometric cell complex;

(ii) D_0 is a d-polytope such that $D_0 = \operatorname{set} \mathscr{C}$, and each face of D_0 is a member of $\mathscr{C}$; and

(iii) $C \cap \operatorname{bd} D_0$ is a member of $\mathscr{C}$, whenever $C \in \mathscr{C}$ (see [6]).

We call D_0 the *basic facet* of the diagram. In a Schlegel-diagram this is the facet, into which the polytope is projected.

The famous Theorem of Steinitz says that every 2-diagram is isomorphic to a Schlegel-diagram of a 3-polytope. So far, no such Theorem for higher dimensions is known. Moreover there are a lot of examples of 3-diagrams, which are not isomorphic to a Schlegel-diagram of a 4-polytope (see [1, 4, 5, 6, 8]).

So, one has to look for additional conditions, a diagram has to satisfy in order to be a Schlegel-diagram. Given a polytope P, a Schlegel-diagram of P may be obtained by projecting P into an arbitrarily chosen facet. This implies that a Schlegel-diagram is invertable: A d-diagram $\mathscr{D}$ is called *invertable* if it may be realized with any of its d-cells serving as the basic facet D_0. Although the diagrams mentioned above turned out not to be invertable, this condition is not sufficient for a diagram to be isomorphic to a Schlegel-diagram. D. Barnette constructed an invertable 3-diagram with 14 vertices, which is not isomorphic to a Schlegel-diagram of a 4-polytope (see [3]).

In Section 1 of this note we present another example of such an invertable 3-diagram, which has only 8 vertices. Our example is smallest possible with respect to the number of vertices, because it is shown in [7] that each d-diagram with less than $d+5$ vertices is isomorphic to a Schlegel-diagram of a $(d+1)$-polytope. Furthermore we modify $\mathscr{D}$ to obtain a diagram $\mathscr{D}'$, which cannot be realized by a polyhedral 3-sphere in E^4.

In the second section the dual structure of $\mathscr{D}$ and $\mathscr{D}'$ will be investigated. Our Notation is according to Grünbaum [6], but we shall abbreviate the convex hull of the points $x_1, \dots, x_n$ by $[x_1, \dots, x_n]$.

1. The diagrams $\mathscr{D}$ and $\mathscr{D}'$

The construction of $\mathscr{D}$

We start with the 3-polytope Q shown in Fig. 1, and choose a point $7 \in \operatorname{int} Q \setminus [2,3,5,6]$. Now $A = [2,3,5,6,7]$ is a doubletetrahedron. The other 3-cells of $\mathscr{D}$ are

$$B=[1,2,3,7], \quad C=[4,5,6,7], \quad D=[1,2,4,5,7],$$
$$E=[1,3,4,6,7], \quad F=[1,2,3,8], \quad G=[4,5,6,8],$$
$$H=[1,2,4,5,8], \quad I=[1,3,4,6,8], \quad K=[2,3,5,8],$$

and

$$L=[3,5,6,8],$$

where 8 is a point in 3-space, from which all 2-faces of Q but one can be seen.

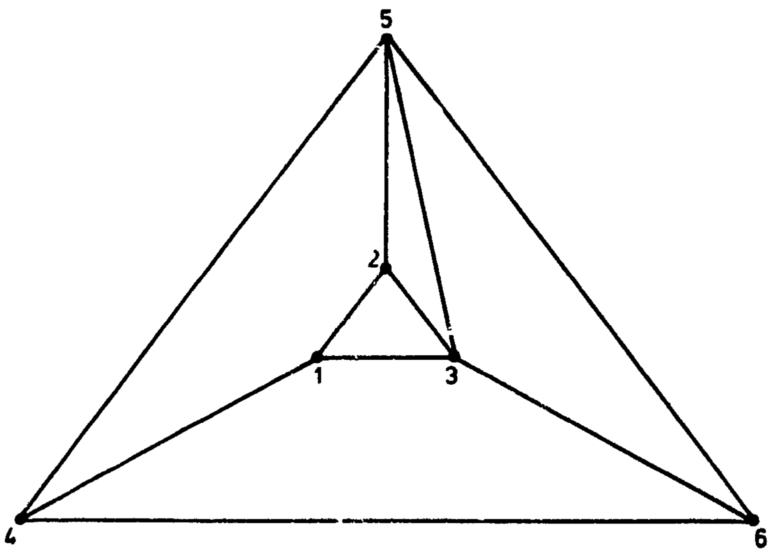

Fig. 1.

For each 2-face of Q there is a realization of Q, for which such a point 8 exists, and it is always possible to choose an appropriate point 7 in the interior of Q. Thus $\mathscr{D}$ is realizable with each of the facets $F, \dots, L$ as the basic facet.

To obtain a realization of $\mathscr{D}$ with one of the facets $B, \dots, E$ as the basic facet, we start with a realization of the polytope Q' shown in Fig. 2, where all 2-faces but $[1,2,3]$, $[4,5,6]$, $[1,2,4,5]$, or $[1,3,4,6]$ respectively can be seen from one point 7 of E^3. Then we replace the faces $[2,3,6]$, $[2,5,6]$, and $[2,6]$ by $[2,3,5]$,

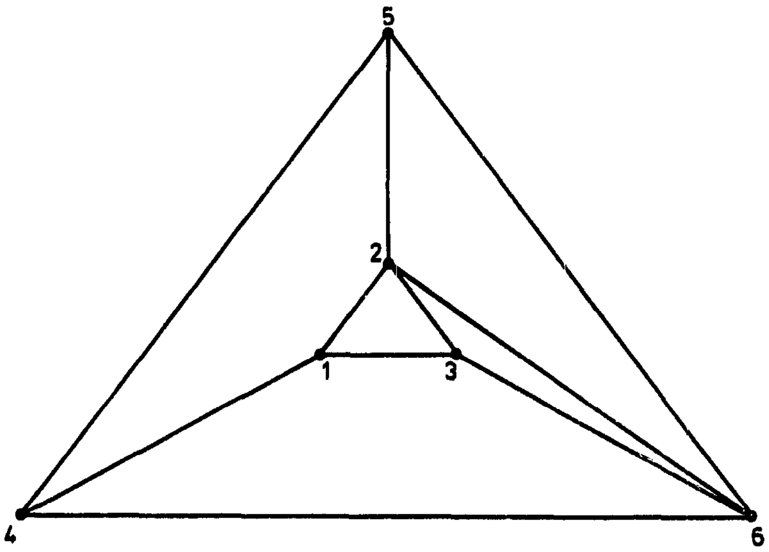

Fig. 2.

[3, 5, 6], and [5, 6], obtaining a 2-sphere, which bounds a non-convex but star-shaped ball $\bar{Q}$. Now we get a realization of $\mathscr{D}$ by choosing $8 \in \operatorname{int} \bar{Q}$ such that all faces of $\bar{Q}$ can be seen from 8.

Finally we realize $\mathscr{D}$ with A as the basic facet by choosing a realization of Q, where all 2-faces but [2, 3, 5] and [3, 5, 6] can be seen from one point 7 of E^3, and taking $8 \in \operatorname{int} Q$. An appropriate realization of Q exists, because the circuit formed by the edges [2, 3], [3, 6], [6, 5], and [5, 2] is a shadow boundary of Q (see [2]). This completes the proof that $\mathscr{D}$ is invertable.

$\mathscr{D}$ contains the boundary complex of Q as a subcomplex S. For any realization of S in euclidian space we have $\dim S = 3$, because all the vertices of S are contained in the two quadrilaterals [1, 2, 4, 5] and [1, 3, 4, 6] sharing the edge [1, 4]. Furthermore the 3-cell A shares two 2-faces [2, 3, 5] and [3, 5, 6] with S, and this implies $\dim \operatorname{aff}\{1, \ldots, 7\} = 3$. So, the assumption that $\mathscr{D}$ is isomorphic to the Schlegel-diagram of a 4-polytope P yields the contradiction $\dim \operatorname{st}(7, P) = 3$.

Remark. There is a refinement of $\mathscr{D}$ without extra edges, which is isomorphic to the Schlegel-diagram of the bypyramid with basis Q. (Just replace A by the two tetrahedra [2, 3, 5, 7] and [3, 5, 6, 7].) On the other hand there exist diagrams, isomorphic to a Schlegel-diagram of a 4-polytope, which have a refinement without extra edges, that is not isomorphic to a Schlegel-diagram. So, the operation "refinement without extra edges" neither preserves the property of being isomorphic to a Schlegel-diagram nor the converse one.

The combinatorial structure of $\mathscr{D}$ may not be realized by the boundary complex of a 4-polytope, but it may be realized by a polyhedral 3-sphere. For this purpose one has only to realize all the cells of $\mathscr{D}$, which are not incident to 8, within a hyperplane H of E^4, and then choose $8 \in E^4 \setminus H$. We shall now change $\mathscr{D}$ into a diagram $\mathscr{D}'$, which cannot even be realized by a polyhedral sphere.

The construction of $\mathscr{D}'$

We start with a realization of $\mathscr{D}$ with $C = [4, 5, 6, 7]$ as the basic facet, and the additional property that the segment [7, 8] meets the triangle [1, 2, 3] in a relatively interior point. Now $\mathfrak{D}'$ is obtained from $\mathfrak{D}$ by replacing the two tetrahedra $[1,2,3,7]$, $[1,2,3,8]$, and the triangle $[1,2,3]$ by the doubletetrahedron $[1,2,3,7,8]$. The assumption that $\mathfrak{D}'$ is realized by a polyhedral 3-sphere $\mathcal{S}$ in euclidian space yields the contradiction $\dim \operatorname{aff} \mathcal{S} = 3$.

2. The dual structure of $\mathfrak{D}$ and $\mathfrak{D}'$

For each polytope P there exists the dual polytope P^*, i.e. a polytope, whose boundary complex may be mapped onto the boundary complex of P by a bijective, inclusion-reversing mapping. So, for each Schlegel-diagram of a polytype P there exists a dual diagram, namely a Schlegel-diagram of P^*. For most of the known 3-diagrams, which are not isomorphic to Schlegel-diagrams, the non-existence of dual diagrams has been proved. In fact it has been shown that not even their 2-skeletons may be realized by plane convex cells in euclidian space (see [4], [8]). We shall now prove the same result for the diagrams $\mathfrak{D}$ and $\mathfrak{D}'$.

Let us assume there is a diagram $\mathfrak{D}^*$, which is dual to the diagram $\mathfrak{D}$ of the last section. Fig. 3 shows the 3-cells of $\mathfrak{D}^*$, corresponding to the vertices 4, 5, and 6 of

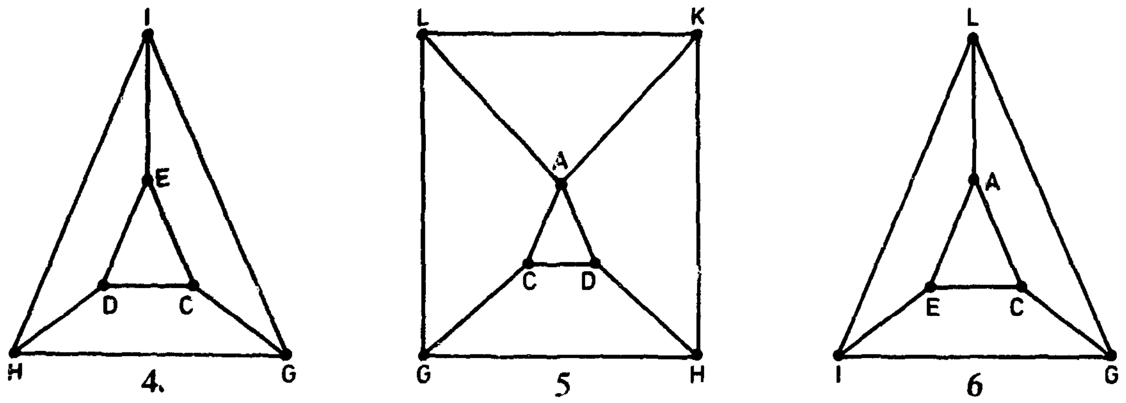

Fig. 3.

$\mathfrak{D}$. (k-faces of $\mathfrak{D}$ and the corresponding $(3-k)$-faces of $\mathfrak{D}^*$ are denoted by the same symbols.) In a triangular prism the affine hulls of the three edges, which are not contained in the boundary of a triangular face, intersect in one point (which may be at infinity). So, considering the faces 4 and 6 of $\mathfrak{D}^*$, we find that the lines $\overline{EI} = \operatorname{aff}\{E, I\}$, $\overline{HD} = \operatorname{aff}\{H, D\}$, $\overline{CG} = \operatorname{aff}\{C, G\}$, and $\overline{AL} = \operatorname{aff}\{A, L\}$ meet in one point P. From this we get a contradiction by looking at face 5 of $\mathfrak{D}^*$. Because P lies on $\overline{HD}$, we have $P \in \operatorname{aff}\{A, D, H, K\}$. On the other hand P is on $\overline{AL}$, and thereby $P = A$. Now $[A, L]$ and $[C, G]$ cannot be edges of $[A, C, G, L]$, because their affine hulls intersect in A. This proves that $\mathfrak{D}^*$ does not exist. $\mathfrak{D}'^*$ also does not exist, because 4, 5, and 6 would be 3-cells of $\mathfrak{D}'^*$ as well.

References

[1] D. Barnette, Diagrams and Schlegel-diagrams, Combinatorial Structures and their Applications (Gordon and Breach, New York, 1970) 1–4.

[2] D. Barnette, Projections of 3-polytopes, Israel J. Math. 8 (1970) 304–308.
[3] D. Barnette, An invertable non-polyhedral diagram, Typescript 1976.
[4] D. Barnette and G. Wegner, A 3-sphere that is not 4-polyheral, Studia Sc. Math. Hung. 6 (1971) 341–346.
[5] U. Betke, Ch. Schulz and J.M. Wills, Bänder und Möbiusbänder in konvexen Polytopen, Abh. Math. Sem. Univ. Hamburg 44 (1975) 249–262.
[6] B. Grünbaum, Convex Polytopes (Interscience Publishers, New York, 1967).
[7] P. Kleinschmidt, Sphären mit wenigen Ecken, Geom. Dedicata 5 (1976) 307–320.
[8] P. Kleinschmidt, Nicht-dualisierbare Sphären, Geom. Dedicata 6 (1977) 511–515.